The Frölicher-Nijenhuis Calculus in Synthetic Differential Geometry

Hirokazu Nishimura
Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki, 305-8571
Japan

November 2, 2008

Abstract

Just as the Jacobi identity of vector fields is a natural consequence of the general Jacobi identity of microcubes in synthetic differential geometry, it is to be shown in this paper that the graded Jacobi identity of the Frölicher-Nijenhuis bracket is also a natural consequence of the general Jacobi identity of microcubes.

1 Introduction

It has long been known that the totality of vector fields on a well-behaved space forms a Lie algebra. Since vector fields and their corresponding derivations can not be identified in synthetic differential geometry, it is by no means direct to establish this fact synthetically. It was Nishimura [9] that noted, behind the Jacobi identity of vector fields, what is to be called the general Jacobi identity of microcubes.

The Frölicher-Nijenhuis bracket, discussed in [1] and [8], is a natural extension of the Lie bracket of vector fields to tangent-vector-valued differential forms. The principal objective in this paper is to derive the graded Jacobi identity for the Frölicher-Nijenhuis bracket from the general Jacobi identity synthetically. The interior derivation and the Lie derivation are discussed in passing.

2 Preliminaries

We assume that the reader is familiar with Lavendhomme's textbook [3] on synthetic differential geometry up to Chapter 5. We denote by D the subset of $\mathbb{R}$ (the extended set of real numbers satisfying the Kock-Lawvere axiom) consisting of elements d of $\mathbb{R}$ with $d^2 = 0$. Given a function $F : D \to \mathbf{E}$ of D into a Euclidean space $\mathbf{E}$, we write $\mathbf{D}F$ for the entity of $\mathbf{E}$ characterized by

$$F(d) = F(0) + d\mathbf{D}F$$

for any $d \in D$.

Given a microlinear space M, we denote M^D by $\mathbf{T}M$. The notion of strong difference $\dot{-}$ was introduced by Kock and Lavendhomme [2] into synthetic differential geometry. The following proposition belongs to the folklore.

Proposition 1 *For any function $f : M \to N$ of microlinear spaces and any $\gamma_1, \gamma_2 \in M^{D^2}$ with $\gamma_1 \mid_{D(2)} = \gamma_2 \mid_{D(2)}$, we have*

$$f \circ \gamma_1 \dot{-} f \circ \gamma_2 = f \circ \left(\gamma_1 \dot{-} \gamma_2 \right)$$

The notion of strong difference $\dot{-}$ can be relativized. Since $M^{D^3} = (M^D)^{D^2}$, microcubes on M can be viewed as microsquares on M^D. According to which D in the right-hand side of $D^3 = D \times D \times D$ appears as the superscript just over M, we get the three relativized strong differences $\underset{i}{\dot{-}}$ $(i = 1, 2, 3)$, for which we have the following general Jacobi identity.

Theorem 2 *Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in M^{D^3}$. As long as the following three expressions are well defined, they sum up only to vanish:*

$$(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$$
$$(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$$
$$(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$$

The theorem was established by Nishimura [9] and has been reproved twice by himself in [10] and [11].

We use the notion of *linear connection* in the sense of Definition 1 in §§5.1 of Lavendhomme [3]. Given a linear connection ∇ on a microlinear space M and a linear connection ∇' on a microlinear space N with a function $f : M \to N$, we say that ∇' *is f-related to* ∇ provided that

$$f \circ \nabla(t_1, t_2) = \nabla'(f \circ t_1, f \circ t_2)$$

for any $t_1, t_2 \in \mathbf{T}M$ with $t_1(0) = t_2(0)$. We will often write $\nabla\gamma$ for $\nabla(t_1, t_2)$, where $t_1 = \gamma(\cdot, 0)$ and $t_2 = \gamma(0, \cdot)$.

We write $\mathbb{S}_n$ for the permutation group of the first n natural numbers, namely, $1, ..., n$. Given $\gamma \in M^{D^n}$ and $\sigma \in \mathbb{S}_n$, we define $\gamma^\sigma \in M^{D^n}$ to be

$$\gamma^\sigma(d_1, ..., d_n) = \gamma(d_{\sigma(1)}, ..., d_{\sigma(n)})$$

for any $(d_1, ..., d_n) \in D^n$. Given $\gamma \in M^{D^n}$, we write $\mathfrak{o}_n(\gamma)$ for $\gamma(0, ..., 0)$.

3 Tangent-Vector-Valued Differential Forms

It is well known in synthetic differential geometry that vector fields can be viewed in three different but essentially equivalent ways, namely, as sections of the tangent bundle, as infinitesimal flows and as infinitesimal transformations, for which the reader is referred to §3.2 of Lavendhomme [3]. These three viewpoints can easily be extended to tangent-vector-valued differential forms. Let M be a microlinear space. The first orthodox viewpoint is to regard tangent-vector-valued differential p-forms on M as mappings $K : M^{D^p} \to M^D$ with $\mathfrak{o}_p(\gamma) = \mathfrak{o}_1(K(\gamma))$ for any $\gamma \in M^{D^p}$ and satisfying the p-homogeneity and the alternating property in the sense of Definition 1 in §4.1 of Lavendhomme [3]. The second viewpoint goes as follows.

Proposition 3 *Tangent-vector-valued differential p-forms on M can be identified with mappings $K : D \times M^{D^p} \to M$ pursuant to the following conditions:*

1. $K(0, \gamma) = \mathfrak{o}_p(\gamma)$ *for any* $\gamma \in M^{D^p}$.
2. $K(\alpha d, \gamma) = K(d, \alpha \underset{i}{\cdot} \gamma)$ *for any* $d \in D$, *any* $\alpha \in \mathbb{R}$, *any* $\gamma \in M^{D^p}$ *and any natural number* i *with* $1 \leq i \leq p$.
3. $K(d, \gamma^\sigma) = K(\varepsilon_\sigma d, \gamma)$ *for any* $d \in D$, *any* $\gamma \in M^{D^p}$ *and any* $\sigma \in \mathbb{S}_p$.

Proof. This follows from the set-theoretical identity

$$(M^D)^{M^{D^p}} = M^{D \times M^{D^p}}$$

The details can safely be left to the reader. ■

Given $\varphi \in M^{M^{D^p}}$ and $\alpha \in \mathbb{R}$, we define $\alpha \underset{i}{\cdot} \varphi \in M^{M^{D^p}}$ $(1 \leq i \leq p)$ to be

$$(\alpha \underset{i}{\cdot} \varphi)(\gamma) = \varphi(\alpha \underset{i}{\cdot} \gamma)$$

for any $\gamma \in M^{D^p}$. Given $\varphi \in M^{M^{D^p}}$ and any $\sigma \in \mathbb{S}_p$, we define $\varphi^\sigma \in M^{M^{D^p}}$ to be

$$\varphi^\sigma(\gamma) = \varphi(\gamma^\sigma)$$

for any $\gamma \in M^{D^p}$. Given $\varphi \in M^{M^{D^p}}$ and $\sigma, \tau \in \mathbb{S}_p$, it is easy to see that

$$\varphi^{\sigma\tau}(\gamma) = \varphi(\gamma^{\sigma\tau}) = \varphi((\gamma^\sigma)^\tau) = \varphi^\tau(\gamma^\sigma) = (\varphi^\tau)^\sigma(\gamma)$$

for any $\gamma \in M^{D^p}$, so that $\varphi^{\sigma\tau} = (\varphi^\tau)^\sigma$. The third viewpoint goes as follows.

Proposition 4 *Tangent-vector-valued differential p-forms on M can be identified with mappings $K : D \to M^{M^{D^p}}$ satisfying the following conditions:*

1. $K_0 = \mathfrak{o}_p$
2. $\alpha \underset{i}{\cdot} K_d = K_{\alpha d}$ *for any* $d \in D$, *any* $\alpha \in \mathbb{R}$ *and any natural number* i *with* $1 \leq i \leq p$.

3. $(K_d)^\sigma = K_{\varepsilon_\sigma d}$ *for any* $d \in D$ *and any* $\sigma \in \mathbb{S}_p$*.*

Proof. This follows from the set-theoretical identity

$$(M^D)^{M^{D^p}} = (M^{M^{D^p}})^D$$

The details can safely be left to the reader. ■

We will use the above three viewpoints on tangent-vector-valued differential forms interchangeably, though we prefer the last one to the preceding two. The following lemma, which will be used in the next section, should be obvious.

Lemma 5 *For any mappings* $K, L : D^2 \to M^{M^{D^p}}$ *with*

$$K(d,0) = L(d,0)$$
$$K(0,d) = L(0,d)$$

for any $d \in D$*, we have*

$$\{(d_1,d_2) \in D^2 \mapsto L(d_1,d_2)^\sigma\} \dot{-}$$
$$\{(d_1,d_2) \in D^2 \mapsto K(d_1,d_2)^\sigma\}$$
$$= d \in D \mapsto ((L \dot{-} K)_d)^\sigma$$

for any $\sigma \in \mathbb{S}_p$*. Similar formulas hold for* $\underset{i}{\dot{-}}$ *(*$i = 1,2,3$*).*

We will write $\Omega^k(M;\mathbf{T}M)$ for the totality of tangent-vector-valued differential k-forms on M. Given $K \in \Omega^k(M;\mathbf{T}M)$, $K' \in \Omega^k(N;\mathbf{T}N)$ and $f : M \to N$, we say that K' is f*-related* to K if we have

$$K'_d(f \circ \gamma) = f(K_d(\gamma))$$

for any $d \in D$ and any $\gamma \in M^D$.

If we drop the condition of the alternating property while keeping the k-homogeneity in the definition of a tangent-vector-valued differential k-form on M, we get the notion of a tangent-vector-valued differential k-semiform on M. We denote by $\widehat{\Omega}^k(M;\mathbf{T}M)$ the totality of tangent-vector-valued differential k-semiforms on M. Given $K \in \widehat{\Omega}^k(M;\mathbf{T}M)$, we define $\mathcal{A}K \in \Omega^k(M;\mathbf{T}M)$ to be

$$\mathcal{A}K(\gamma) = \sum_{\sigma \in \mathbb{S}_k} \varepsilon_\sigma K(\gamma^\sigma)$$

for any $\gamma \in M^{D^k}$, where ε_σ is the sign of σ. We write $\mathcal{A}_{p,q}K$ and $\mathcal{A}_{p,q,r}K$ for $(1/p!q!)\mathcal{A}K$ and $(1/p!q!r!)\mathcal{A}K$ respectively.

4 Interior Derivations

Given $K \in \Omega^{k+1}(M; \mathbf{T}M)$ and $L \in \Omega^{l}(M; \mathbf{T}M)$, we define $\widehat{\mathbf{i}}_K L \in \widehat{\Omega}^{k+l}(M; \mathbf{T}M)$ to be

$$\begin{aligned}(\widehat{\mathbf{i}}_K L)(\gamma) = L\{&(e_1, ..., e_l) \in D^l \mapsto K_{e_1}((d_1, ...d_{k+1}) \in D^{k+1} \\ &\mapsto \gamma(d_1, ..., d_{k+1}, e_2, ..., e_l))\}\end{aligned}$$

for any $\gamma \in M^{D^{k+l}}$. Obviously we have to verify that

Proposition 6 *We have*

$$\widehat{\mathbf{i}}_K L \in \widehat{\Omega}^{k+l}(M; \mathbf{T}M)$$

Proof. Let e be an arbitrary element of D with $\alpha \in \mathbb{R}$. For $1 \leq i \leq k+1$, we have

$$\begin{aligned}
&(\widehat{\mathbf{i}}_K L)_e(\alpha \underset{i}{\cdot} \gamma) \\
&= L_e\{(e_1, ..., e_l) \in D^l \mapsto \\
K_{e_1}((d_1, ...d_{k+1}) \in D^{k+1} &\mapsto \gamma(d_1, ..., , \alpha d_i, ..., d_{k+1}, e_2, ..., e_l))\} \\
&= L_e\{(e_1, ..., e_l) \in D^l \mapsto \\
K_{\alpha e_1}((d_1, ...d_{k+1}) \in D^{k+1} &\mapsto \gamma(d_1, ..., , d_i, ..., d_{k+1}, e_2, ..., e_l))\} \\
&= L_{\alpha e}\{(e_1, ..., e_l) \in D^l \mapsto \\
K_{e_1}((d_1, ...d_{k+1}) \in D^{k+1} &\mapsto \gamma(d_1, ..., , d_i, ..., d_{k+1}, e_2, ..., e_l))\} \\
&= (\widehat{\mathbf{i}}_K L)_{\alpha e}(\gamma)
\end{aligned}$$

while the case of $k+2 \leq i \leq k+l$ can safely be left to the reader. ■

Given $K \in \Omega^{k+1}(M; \mathbf{T}M)$ and $L \in \Omega^{l}(M; \mathbf{T}M)$, we define $\mathbf{i}_K L \in \Omega^{k+l}(M; \mathbf{T}M)$ to be

$$\mathcal{A}_{k+1,l-1}\left(\widehat{\mathbf{i}}_K L\right)$$

Proposition 7 *Let $f : M \to N$ be a mapping. Let us suppose that $K' \in \Omega^{k+1}(N; \mathbf{T}N)$ is f-related to $K \in \Omega^{k+1}(M; \mathbf{T}M)$ and that $L' \in \Omega^{l}(N; \mathbf{T}N)$ is f-related to $L \in \Omega^{l}(M; \mathbf{T}M)$. Then $\mathbf{i}_{K'} L'$ is f-related to $\mathbf{i}_K L$.*

Proof. It suffices to show that $\widehat{\mathbf{i}}_{K'}L'$ is f-related to $\widehat{\mathbf{i}}_K L$. Let $d \in D$ and $\gamma \in M^{D^{k+l}}$. Then we have

$$\begin{aligned}
& f((\widehat{\mathbf{i}}_K L)_e(\gamma)) \\
& = f[L_e\{(e_1,...,e_l) \in D^l \mapsto \\
K_{e_1}((d_1,...d_{k+1}) & \in D^{k+1} \mapsto \gamma(d_1,...,d_{k+1},e_2,...,e_l))\}] \\
& = L'_e[(e_1,...,e_l) \in D^l \mapsto \\
f\{K_{e_1}((d_1,...d_{k+1}) & \in D^{k+1} \mapsto \gamma(d_1,...,d_{k+1},e_2,...,e_l))\}] \\
& = L'_e[(e_1,...,e_l) \in D^l \mapsto K'_{d_1}((f \circ \gamma)(\cdot_1,...,\cdot_{k+1},d_2,...,d_l)) \\
K_{e_1}\{(d_1,...d_{k+1}) & \in D^{k+1} \mapsto (f \circ \gamma)(d_1,...,d_{k+1},e_2,...,e_l)\}] \\
& = (\widehat{\mathbf{i}}_{K'}L')_e(f \circ \gamma)
\end{aligned}$$

which completes the proof. ■

5 The Frölicher-Nijenhuis Bracket

Given $\varphi \in M^{M^{D^p}}$ and $\psi \in M^{M^{D^q}}$, we define $\psi * \varphi \in M^{M^{D^{p+q}}}$ to be

$$\begin{aligned}
& \psi * \varphi(\gamma) \\
& = \psi\{(e_1,...,e_q) \in D^q \mapsto \\
\varphi((d_1,...,d_p) & \in D^p \mapsto \gamma(d_1,...,d_p,e_1,...,e_q))\}
\end{aligned}$$

for any $\gamma \in M^{D^{p+q}}$. Given two tangent-vector-valued differential forms $K : D \to M^{M^{D^p}}$ and $L : D \to M^{M^{D^q}}$, we define a mapping $L * K : D^2 \to M^{M^{D^{p+q}}}$ to be

$$(L * K)(d_1, d_2) = L_{d_2} * K_{d_1}$$

for any $(d_1, d_2) \in D^2$. The following lemma should be obvious.

Lemma 8 *Given two tangent-vector-valued differential forms $K : D \to M^{M^{D^p}}$ and $L : D \to M^{M^{D^q}}$ with $\sigma = \begin{pmatrix} 1 & ... & q & q+1 & ... & p+q \\ p+1 & ... & p+q & 1 & ... & p \end{pmatrix} \in \mathbb{S}_{p+q}$, we have*

1. *$(L * K)(d, 0) = ((K * L)(0, d))^\sigma$ for any $d \in D$.*
2. *$(L * K)(0, d) = ((K * L)(d, 0))^\sigma$ for any $d \in D$.*

We continue to use the notation of the above lemma for a while. We denote by $K \widetilde{*} L$ the mapping $(d_1, d_2) \in D^2 \mapsto ((K * L)(d_2, d_1))^\sigma \in M^{M^{D^{p+q}}}$. We are thus entitled by the above lemma to define $\lfloor K, L \rfloor \in (M^{M^{D^{p+q}}})^D$ to be

$$L * K \stackrel{\cdot}{-} K \widetilde{*} L$$

Lemma 9 *The mapping* $\lfloor K, L \rfloor : D \to M^{M^{D^{p+q}}}$ *satisfies the following conditions:*

1. $\lfloor K, L \rfloor_0 = \mathfrak{o}_p$
2. $\alpha \underset{i}{\cdot} \lfloor K, L \rfloor_d = \lfloor K, L \rfloor_{\alpha d}$ *for any* $d \in D$*, any* $\alpha \in \mathbb{R}$ *and any natural number* i *with* $1 \leq i \leq p+q$.

Proof. The first condition should be obvious. To see the second condition, we note that

1. $\alpha \underset{i}{\cdot} (L {*} K)(d_1, d_2) = (L {*} K)(\alpha d_1, d_2)$ and $\alpha \underset{i}{\cdot} (K \widetilde{*} L)(d_1, d_2) = (K \widetilde{*} L)(\alpha d_1, d_2)$ for any natural number i with $1 \leq i \leq p$.
2. $\alpha \underset{i}{\cdot} (L {*} K)(d_1, d_2) = (L {*} K)(d_1, \alpha d_2)$ and $\alpha \underset{i}{\cdot} (K \widetilde{*} L)(d_1, d_2) = (K \widetilde{*} L)(d_1, \alpha d_2)$ for any natural number i with $p+1 \leq i \leq p+q$.

Therefore the second condition follows by Proposition 5 in §3.4 of Lavendhomme [3] from the first property in case of $1 \leq i \leq p$ and from the second property in case of $p+1 \leq i \leq p+q$. ■

Lemma 10 *Given three tangent-vector-valued differential forms* $K_1 : D \to M^{M^{D^p}}$, $K_2 : D \to M^{M^{D^q}}$ *and* $K_3 : D \to M^{M^{D^r}}$*, we have*

$$\mathcal{A}_{p,q+r}(\lfloor K_1, \mathcal{A}_{q,r}(\lfloor K_2, K_3 \rfloor) \rfloor) = \mathcal{A}_{p,q,r}(\lfloor K_1, \lfloor K_2, K_3 \rfloor \rfloor)$$

Proof. By the same token as in the familiar associativity of wedge products in differential forms. ■

We are going to define the *Frölicher-Nijenhuis bracket* $\lceil K, L \rceil$ to be

$$\lceil K, L \rceil = \mathcal{A}_{p,q}(\lfloor K, L \rfloor)$$

which is undoubtedly a tangent-vector-valued differential $(p+q)$-form.

Theorem 11 *The following two properties hold for the Frölicher-Nijenhuis bracket:*

1. *We have*
$$\lceil K, L \rceil = -(-1)^{pq} \lceil L, K \rceil$$
for any two tangent-vector-valued differential forms $K : D \to M^{M^{D^p}}$ *and* $L : D \to M^{M^{D^q}}$.
2. *We have*
$$\lceil K_1, \lceil K_2, K_3 \rceil \rceil + (-1)^{p(q+r)} \lceil K_2, \lceil K_3, K_1 \rceil \rceil + (-1)^{r(p+q)} \lceil K_3, \lceil K_1, K_2 \rceil \rceil = 0$$
for any three tangent-vector-valued differential forms $K_1 : D \to M^{M^{D^p}}$, $K_2 : D \to M^{M^{D^q}}$ *and* $K_3 : D \to M^{M^{D^r}}$.

Proof. In order to see the first property, it suffices to note that

$$\begin{aligned} (L*K)(d_1,d_2)^\sigma &= (L\widetilde{*}K)(d_2,d_1) \\ (K\widetilde{*}L)(d_1,d_2)^\sigma &= (K*L)(d_2,d_1) \end{aligned}$$

from which it follows Propositions 4 and 6 in §3.4 of Lavendhomme [3] and Lemma 5 that

$$\begin{aligned} &\lceil K,L \rceil \\ &= \mathcal{A}_{p,q}(\lfloor K,L \rfloor) \\ &= \frac{1}{p!q!}\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau\varepsilon_\sigma\{d\in D\mapsto ((L*K \dot{-} K\widetilde{*}L)_d)^{\tau\sigma}\} \\ &= \frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau[\{(d_1,d_2)\in D^2\mapsto (L*K)(d_1,d_2)^{\tau\sigma}\}\dot{-} \\ &\{(d_1,d_2)\in D^2\mapsto (K\widetilde{*}L)(d_1,d_2)^{\tau\sigma}\}] \\ &= \frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau[\{(d_1,d_2)\in D^2\mapsto ((L*K)(d_1,d_2)^{\sigma})^\tau\}\dot{-} \\ &\{(d_1,d_2)\in D^2\mapsto ((K\widetilde{*}L)(d_1,d_2)^{\sigma})^\tau\}] \\ &= \frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau[\{(d_1,d_2)\in D^2\mapsto (L\widetilde{*}K)(d_2,d_1)^{\tau}\}\dot{-} \\ &\{(d_1,d_2)\in D^2\mapsto (K*L)(d_2,d_1)^{\tau}\}] \\ &= \frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau\{d\in D\mapsto ((L\widetilde{*}K \dot{-} K*L)_d)^{\tau}\} \\ &= -\frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau\{d\in D\mapsto ((K*L \dot{-} L\widetilde{*}K)_d)^{\tau}\} \\ &= -\frac{1}{p!q!}\varepsilon_\sigma\sum_{\tau\in\mathbb{S}_{p+q}} \varepsilon_\tau\{d\in D\mapsto (\lfloor L,K \rfloor_d)^{\tau}\} \\ &= -\varepsilon_\sigma \lceil L,K \rceil \end{aligned}$$

Since it is easy to see that $\varepsilon_\sigma = (-1)^{pq}$, the desired first property follows at once. In order to see the second property, we first define six mappings

$\varphi_{123}, \varphi_{132}, \varphi_{213}, \varphi_{231}, \varphi_{312}, \varphi_{321} : D^3 \to M^{M^{D^{p+q+r}}}$ to be

$$\varphi_{123} = (d_1, d_2, d_3) \in D^3 \mapsto (K_3)_{d_3} * (K_2)_{d_2} * (K_1)_{d_1} \in M^{M^{D^{p+q+r}}}$$
$$\varphi_{132} = (d_1, d_2, d_3) \in D^3 \mapsto ((K_2)_{d_2} * (K_3)_{d_3} * (K_1)_{d_1})^{\sigma_{132}} \in M^{M^{D^{p+q+r}}}$$
$$\varphi_{213} = (d_1, d_2, d_3) \in D^3 \mapsto ((K_3)_{d_3} * (K_1)_{d_1} * (K_2)_{d_2})^{\sigma_{213}} \in M^{M^{D^{p+q+r}}}$$
$$\varphi_{231} = (d_1, d_2, d_3) \in D^3 \mapsto ((K_1)_{d_1} * (K_3)_{d_3} * (K_2)_{d_2})^{\sigma_{231}} \in M^{M^{D^{p+q+r}}}$$
$$\varphi_{312} = (d_1, d_2, d_3) \in D^3 \mapsto ((K_2)_{d_2} * (K_1)_{d_1} * (K_3)_{d_3})^{\sigma_{312}} \in M^{M^{D^{p+q+r}}}$$
$$\varphi_{321} = (d_1, d_2, d_3) \in D^3 \mapsto ((K_1)_{d_1} * (K_2)_{d_2} * (K_3)_{d_3})^{\sigma_{321}} \in M^{M^{D^{p+q+r}}}$$

where

$$\sigma_{132} = \begin{pmatrix} 1 & ... & p & p+1 & ... & p+r & p+r+1 & ... & p+q+r \\ 1 & ... & p & p+q+1 & ... & p+q+r & p+1 & ... & p+q \end{pmatrix}$$
$$\sigma_{213} = \begin{pmatrix} 1 & ... & q & q+1 & ... & p+q & p+q+1 & ... & p+q+r \\ p+1 & ... & p+q & 1 & ... & p & p+q+1 & ... & p+q+r \end{pmatrix}$$
$$\sigma_{231} = \begin{pmatrix} 1 & ... & q & q+1 & ... & q+r & q+r+1 & ... & p+q+r \\ p+1 & ... & p+q & p+q+1 & ... & p+q+r & 1 & ... & p \end{pmatrix}$$
$$\sigma_{312} = \begin{pmatrix} 1 & ... & r & r+1 & ... & p+r & p+r+1 & ... & p+q+r \\ p+q+1 & ... & p+q+r & 1 & ... & p & p+1 & ... & p+q \end{pmatrix}$$
$$\sigma_{321} = \begin{pmatrix} 1 & ... & r & r+1 & ... & q+r & q+r+1 & ... & p+q+r \\ p+q+1 & ... & p+q+r & p+1 & ... & p+q & 1 & ... & p \end{pmatrix}$$

Now we have

$$\begin{aligned} &\lceil K_1, \lceil K_2, K_3 \rceil \rceil \\ &= \mathcal{A}_{p,q+r}(\lfloor K_1, \mathcal{A}_{q,r}(\lfloor K_2, K_3 \rfloor) \rfloor) \\ &= \mathcal{A}_{p,q,r}(\lfloor K_1, \lfloor K_2, K_3 \rfloor \rfloor) \\ &= \mathcal{A}_{p,q,r}((\varphi_{123} \overset{\cdot}{\underset{1}{-}} \varphi_{132}) \overset{\cdot}{-} (\varphi_{231} \overset{\cdot}{\underset{1}{-}} \varphi_{321})) \end{aligned}$$

Let $\rho_1 \in \mathbb{S}_{p+q+r}$ be σ_{231}, for which we have $\varepsilon_{\rho_1} = (-1)^{p(q+r)}$. Let $\varphi_{231}^2, \varphi_{213}^2, \varphi_{312}^2, \varphi_{132}^2 : D^3 \mapsto M^{M^{D^{p+q+r}}}$ be mapping

$$\varphi_{231}^2 = (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{231}(d_1, d_2, d_3)^{\rho_1}$$
$$\varphi_{213}^2 = (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{213}(d_1, d_2, d_3)^{\rho_1}$$
$$\varphi_{312}^2 = (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{312}(d_1, d_2, d_3)^{\rho_1}$$
$$\varphi_{132}^2 = (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{132}(d_1, d_2, d_3)^{\rho_1}$$

Now we have

$$
\begin{aligned}
&\lceil K_2, \lceil K_3, K_1 \rceil \rceil \\
&= \mathcal{A}_{q,r+p}(\lfloor K_2, \mathcal{A}_{r,p}(\lfloor K_3, K_1 \rfloor) \rfloor) \\
&= \mathcal{A}_{q,r,p}(\lfloor K_2, \lfloor K_3, K_1 \rfloor \rfloor) \\
&= \mathcal{A}_{q,r,p}((\varphi^2_{231} \underset{2}{\dot{-}} \varphi^2_{213}) \dot{-} (\varphi^2_{312} \underset{2}{\dot{-}} \varphi^2_{132})) \\
&= \frac{1}{p!q!r!} \sum_{\tau \in \mathbb{S}_{p+q+r}} \varepsilon_\tau \{ d \in D \mapsto (((\varphi_{231} \underset{2}{\dot{-}} \varphi_{213}) \dot{-} (\varphi_{312} \underset{2}{\dot{-}} \varphi_{132}))_d)^{\tau\rho_1} \} \\
&= \frac{1}{p!q!r!} \varepsilon_{\rho_1} \sum_{\tau \in \mathbb{S}_{p+q+r}} \varepsilon_\tau \varepsilon_{\rho_1} \{ d \in D \mapsto (((\varphi_{231} \underset{2}{\dot{-}} \varphi_{213}) \dot{-} (\varphi_{312} \underset{2}{\dot{-}} \varphi_{132}))_d)^{\tau\rho_1} \} \\
&= \varepsilon_{\rho_1} \mathcal{A}_{q,r,p}((\varphi_{231} \underset{2}{\dot{-}} \varphi_{213}) \dot{-} (\varphi_{312} \underset{2}{\dot{-}} \varphi_{132}))
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(-1)^{p(q+r)} \lceil K_2, \lceil K_3, K_1 \rceil \rceil \\
&= \mathcal{A}_{q,r,p}((\varphi_{231} \underset{2}{\dot{-}} \varphi_{213}) \dot{-} (\varphi_{312} \underset{2}{\dot{-}} \varphi_{132}))
\end{aligned}
$$

Let $\rho_2 \in \mathbb{S}_{p+q+r}$ be σ_{312}, for which we have $\varepsilon_{\rho_2} = (-1)^{r(p+q)}$. Let $\varphi^3_{312}, \varphi^3_{321}, \varphi^3_{123}, \varphi^3_{213} : D^3 \mapsto M^{M^{D^{p+q+r}}}$ be mappings

$$
\begin{aligned}
\varphi^3_{312} &= (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{312}(d_1, d_2, d_3)^{\rho_2} \\
\varphi^3_{321} &= (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{321}(d_1, d_2, d_3)^{\rho_2} \\
\varphi^3_{123} &= (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{123}(d_1, d_2, d_3)^{\rho_2} \\
\varphi^3_{213} &= (d_1, d_2, d_3) \in D^3 \mapsto \varphi_{213}(d_1, d_2, d_3)^{\rho_2}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
&\lceil K_3, \lceil K_1, K_2 \rceil \rceil \\
&= \mathcal{A}_{r,p+q}(\lfloor K_3, \mathcal{A}_{p,q}(\lfloor K_1, K_2 \rfloor) \rfloor) \\
&= \mathcal{A}_{r,p,q}(\lfloor K_3, \lfloor K_1, K_2 \rfloor \rfloor) \\
&= \mathcal{A}_{r,p,q}((\varphi^3_{312} \underset{3}{\dot{-}} \varphi^3_{321}) \dot{-} (\varphi^3_{123} \underset{3}{\dot{-}} \varphi^3_{213})) \\
&= \frac{1}{p!q!r!} \sum_{\tau \in \mathbb{S}_{p+q+r}} \varepsilon_\tau \{ d \in D \mapsto (((\varphi_{312} \underset{3}{\dot{-}} \varphi_{321}) \dot{-} (\varphi_{123} \underset{3}{\dot{-}} \varphi_{213}))_d)^{\tau\rho_2} \} \\
&= \frac{1}{p!q!r!} \varepsilon_{\rho_2} \sum_{\tau \in \mathbb{S}_{p+q+r}} \varepsilon_\tau \varepsilon_{\rho_2} \{ d \in D \mapsto (((\varphi_{312} \underset{3}{\dot{-}} \varphi_{321}) \dot{-} (\varphi_{123} \underset{3}{\dot{-}} \varphi_{213}))_d)^{\tau\rho_2} \} \\
&= \varepsilon_{\rho_2} \mathcal{A}_{r,p,q}((\varphi_{312} \underset{3}{\dot{-}} \varphi_{321}) \dot{-} (\varphi_{123} \underset{3}{\dot{-}} \varphi_{213}))
\end{aligned}
$$

which implies that

$$\begin{aligned}&(-1)^{r(p+q)}\left\lceil K_3,\left\lceil K_1,K_2\right\rceil\right\rceil\\&=\mathcal{A}_{r,p,q}((\varphi_{312}\underset{3}{\dot{-}}\varphi_{321})\dot{-}(\varphi_{123}\underset{3}{\dot{-}}\varphi_{213}))\end{aligned}$$

Therefore we have

$$\begin{aligned}&\left\lceil K_1,\left\lceil K_2,K_3\right\rceil\right\rceil+(-1)^{p(q+r)}\left\lceil K_2,\left\lceil K_3,K_1\right\rceil\right\rceil+(-1)^{r(p+q)}\left\lceil K_3,\left\lceil K_1,K_2\right\rceil\right\rceil\\&=\mathcal{A}_{p,q,r}((\varphi_{123}\underset{1}{\dot{-}}\varphi_{132})\dot{-}(\varphi_{231}\underset{1}{\dot{-}}\varphi_{321}))+\\&\mathcal{A}_{q,r,p}((\varphi_{231}\underset{2}{\dot{-}}\varphi_{213})\dot{-}(\varphi_{312}\underset{2}{\dot{-}}\varphi_{132}))+\\&\mathcal{A}_{r,p,q}((\varphi_{312}\underset{3}{\dot{-}}\varphi_{321})\dot{-}(\varphi_{123}\underset{3}{\dot{-}}\varphi_{213}))\\&=\frac{1}{p!q!r!}\mathcal{A}(\{(\varphi_{123}\underset{1}{\dot{-}}\varphi_{132})\dot{-}(\varphi_{231}\underset{1}{\dot{-}}\varphi_{321})\}+\{(\varphi_{231}\underset{2}{\dot{-}}\varphi_{213})\dot{-}(\varphi_{312}\underset{2}{\dot{-}}\varphi_{132})\}\\&+\{(\varphi_{312}\underset{3}{\dot{-}}\varphi_{321})\dot{-}(\varphi_{123}\underset{3}{\dot{-}}\varphi_{213})\})\\&=0\quad\text{[by the general Jacobi identity]}\end{aligned}$$

■

Now we are going to show the naturality of the Frölicher-Nijenhuis bracket. Let $f:M\rightarrow N$ be a function of microlinear spaces.

Lemma 12 *If tangent-vector-valued differential forms $K':D\rightarrow N^{N^{D^p}}$ and $L':D\rightarrow N^{N^{D^q}}$ are f-related to tangent-vector-valued differential forms $K:D\rightarrow M^{M^{D^p}}$ and $L:D\rightarrow M^{M^{D^q}}$ respectively, then we have*

$$\begin{aligned}f\circ((L*K)(\gamma))&=(L'*K')(f\circ\gamma)\\f\circ((K\widetilde{*}L)(\gamma))&=(K'\widetilde{*}L')(f\circ\gamma)\end{aligned}$$

for any $\gamma\in M^{D^{p+q}}$.

Proof. For the first identity, we have

$$\begin{aligned}&f\circ((L*K)(\gamma))\\&=f\circ[(d,d')\in D^2\mapsto L_{d'}\{(e_1,...,e_q)\in D^q\mapsto\\K_d((d_1,...,d_p)&\in D^p\mapsto\gamma(d_1,...,d_p,e_1,...,e_q))\}]\\&=(d,d')\in D^2\mapsto f[L_{d'}\{(e_1,...,e_q)\in D^q\mapsto\\K_d((d_1,...,d_p)&\in D^p\mapsto\gamma(d_1,...,d_p,e_1,...,e_q))\}]\\&=(d,d')\in D^2\mapsto L'_{d'}[(e_1,...,e_q)\in D^q\mapsto\\f\{K_d((d_1,...,d_p)&\in D^p\mapsto\gamma(d_1,...,d_p,e_1,...,e_q))\}]\\&=(d,d')\in D^2\mapsto L'_{d'}[(e_1,...,e_q)\in D^q\mapsto\\\{K_d((d_1,...,d_p)&\in D^p\mapsto(f\circ\gamma)(d_1,...,d_p,e_1,...,e_q))\}]\\&=(L'*K')(f\circ\gamma)\end{aligned}$$

The second formula can be established by the same token. ■

Proposition 13 *Under the same assumption and notation as in the above lemma, $\lceil K', L' \rceil$ is f-related to $\lceil K, L \rceil$.*

Proof. It suffices to show that $\lfloor K', L' \rfloor$ is f-related to $\lfloor K, L \rfloor$, which follows from the following calculation:

$$\begin{aligned}
& f \circ (\lfloor K, L \rfloor (\gamma)) \\
& = f \circ \{(L * K)(\gamma) \dot{-} (K \widetilde{*} L)(\gamma)\} \\
& = f \circ ((L * K)(\gamma)) \dot{-} f \circ ((K \widetilde{*} L)(\gamma)) \\
& = (L' * K')(f \circ \gamma) \dot{-} (K' \widetilde{*} L')(f \circ \gamma) \\
& = \lfloor K', L' \rfloor (f \circ \gamma)
\end{aligned}$$

for any $\gamma \in M^{D^{p+q}}$. ■

6 Lie Derivations

Let $K \in \Omega^k(M; \mathbf{T}M)$ and $L \in \Omega^l(M; \mathbf{T}M)$. Let ∇ be a linear connection on M. It is easy to see that

Lemma 14 *We have*

$$\begin{aligned}
(L * K)(d, 0) = \nabla(L * K)(d, 0) \\
(L * K)(0, d) = \nabla(L * K)(0, d)
\end{aligned}$$

for any $d \in D$.

Now we define $\widehat{\mathbf{L}}_K^{\nabla} L \in \widehat{\Omega}^{k+l}(M; \mathbf{T}M)$ to be

$$\widehat{\mathbf{L}}_K^{\nabla} L(\gamma) = (L * K)(\gamma) \dot{-} \nabla((L * K)(\gamma))$$

for any $\gamma \in M^{D^{k+l}}$. Indeed we have to verify that

Lemma 15 *We have*

$$\widehat{\mathbf{L}}_K^{\nabla} L(\alpha \underset{i}{\cdot} \gamma) = \alpha \left(\widehat{\mathbf{L}}_K^{\nabla} L(\gamma) \right)$$

for any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq i \leq k + l$.

Proof. By the same token as in the proof of Lemma 9. ■

Proposition 16 *With the above notation, we have*

$$\widehat{\mathbf{L}}_K^{\nabla} L(\gamma)$$
$$= \mathbf{D}[e \in D \mapsto \mathbf{q}_{(t,e)}[L\{(e_1, ..., e_l) \in D^l \mapsto K_e((d_1, ...d_k) \in D^k \mapsto \gamma(d_1, ..., d_k, e_1, ..., e_l))\}]]$$

for any $\gamma \in M^{D^{k+l}}$*, where* $t \in M^D$ *is the mapping* $d \in D \mapsto K_d((d_1, ...d_k) \in D^k \mapsto \gamma(d_1, ..., d_k, 0, ..., 0)) \in M$.

Proof. By Propositions 3 and 7 in §§5.2 of Lavendhomme [3]. ■

We define $\mathbf{L}_K^{\nabla} L \in \Omega^{k+l}(M; \mathbf{T}M)$ to be

$$\mathbf{L}_K^{\nabla} L = \mathcal{A}_{k,l}\left(\widehat{\mathbf{L}}_K^{\nabla} L\right)$$

Proposition 17 *Continuing with the above notation and assuming that the linear connection* ∇ *is symmetric, we have*

$$\lceil K, L \rceil = \mathbf{L}_K^{\nabla} L - (-1)^{kl} \mathbf{L}_L^{\nabla} K$$

Proof. It suffices to show that

$$\lfloor K, L \rfloor = \widehat{\mathbf{L}}_K^{\nabla} L - \left(\widehat{\mathbf{L}}_L^{\nabla} K\right)^{\sigma}$$

with $\sigma = \begin{pmatrix} 1 & ... & k & k+1 & ... & k+l \\ k+1 & ... & k+l & 1 & ... & k \end{pmatrix} \in \mathbb{S}_{k+l}$, which follows by the same token as in Proposition 3 in §§5.3 of Lavendhomme [3]. ■

Finally we are going to establish the naturality of Lie derivations. Let $f : M \to N$ be a function of microlinear spaces with a linear connection ∇' on N being f-related to the linear connection ∇ on M.

Lemma 18 *Let* $K' \in \Omega^k(N; \mathbf{T}N)$ *and* $L' \in \Omega^l(N; \mathbf{T}N)$ *be* f*-related to* $K \in \Omega^k(M; \mathbf{T}M)$ *and* $L \in \Omega^l(M; \mathbf{T}M)$ *respectively. Then we have*

$$f \circ (\nabla(L * K)(\gamma)) = \nabla'(L' * K')(f \circ \gamma)$$

for any $\gamma \in M^{D^{k+l}}$.

Proof. By the same token as in the proof of Lemma 12. ■

Proposition 19 *Under the same assumption and notation as in the above lemma,* $\mathbf{L}_{K'}^{\nabla'} L'$ *is* f*-related to* $\mathbf{L}_K^{\nabla} L$.

Proof. By the same token as in the proof of Proposition 13. ■

References

[1] Frölicher, A. and Nijenhuis, A.:Theory of vector-valued differential forms, Part I, Indagationes Math., **18** (1956), 338-359.

[2] Kock, A. and Lavendhomme, R.:Strong infinitesimal linearity, with applications to strong difference and affine connections, Cahiers de Topologie et Geométrie Differéntielle, **25** (1984), 311-324.

[3] Lavendhomme, R.: Basic Concepts of Synthetic Differential Geometry, Kluwer, Dordrecht, 1996.

[4] Michor, Peter W.:Remarks on the Frölicher-Nijenhuis bracket, Proceedings of the Conference on Differential Geometry and its Applications, Brno 1986, D. Reidel, 1987, pp.197-220.

[5] Michor, Peter W.:Topics in Differential Geometry, American Mathematical Society, Providence, Rhode Islands, 2008.

[6] Minguez, M.C.:Wedge products of forms in synthetic differential geometry, Cahiers de Topologie et Geométrie Differéntielle, **29** (1988), 59-66.

[7] Minguez, M.C.:Some combinatorial calculus on Lie derivatives, Cahiers de Topologie et Geométrie Differéntielle, **29** (1988), 241-247.

[8] Nijenhuis, A.:Jacobi type identities for bilinear differential concomitants of certain tensor fields I, Indagationes Math., **17** (1955), 390-403.

[9] Nishimura, H.:Theory of microcubes, International Journal of Theoretical Physics, **36** (1997), 1099-1131.

[10] Nishimura, H.:General Jacobi identity revisited, International Journal of Theoretical Physics, **38** (1999), 2163-2174.

[11] Nishimura, H.:General Jacobi identity revisited again, International Journal of Theoretical Physics, **46** (2007), 2843-2862.